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GENERAL SOLUTION FOR POST-FRAME ROOF DIAPHRAGM DEFLECTIONS

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Summary:

This paper derives single equations for calculating the eave deflections and shears of a laterally loaded post-frame building. It uses the analogy of the roof diaphragm as a continuously loaded beam, where the applied load decreases as the eave deflections increase. This method can accommodate non-rigid end walls, but is limited to uniform diaphragms and frames. Examples are presented.

Keywords:
diaphragms, frames, deflections, shear

GENERAL SOLUTION - ROOF DEFLECTIONS

A uniform roof diaphragm can be modeled as a continuously loaded beam. In conventional stud construction, the studs resist no load as they deflect at the eave, and the roof diaphragm carries the entire eave load, +*w*. In contrast, post frames do resist lateral eave loads in direct proportion to the eave deflection, *y*. Thus the lateral load is shared among the frames and roof diaphragm. This is the key to understanding the method this author proposes: as the deflections at the eave increase, the loads to the roof diaphragm **decrease** and the loads to the frames **increase**. (Figure 1)

The sum of the resistances of the post-frames and diaphragm is equal and opposite to the eave load, *w*. This can be written as:

$$q + r = -w \qquad (1)$$

Where

- $-w$ = the total resistance of the frames and diaphragm at the eave (+*w* = the total wind load along the eave)
- q = the resistance of the roof diaphragm
- r = the resistance of the post-frames

The resistance of the post frames can be expressed as:

$$r = -ky \qquad (2)$$

Where

- k = the lateral stiffness of the frames in pounds per inch of frame deflection at the eave divided by the frame spacing
- y = the lateral deflections in inches.

By substituting (2) into (1) and rearranging:

$$q = ky - w \qquad (3)$$

Since *q* is numerically equal to that portion of the eave load resisted by the diaphragm, it can be applied to the analog beam as a load to predict deflections and shears.

We can simplify the solution if we neglect diaphragm deflections due to bending as small compared to the shear deflections. The differential equation for diaphragm deflection can then be written as (Luttrel, 1987, p. AIII-12, Timoshenko, p. 202):

$$y'' = \frac{q}{C} \qquad (4)$$

Where:

- y'' = the second derivative of *y* with respect to *x*, the coordinate along the beam axis
- C = the stiffness of the diaphragm in pounds

By substituting (3) into (4) and rearranging terms:

$$y'' - \left(\frac{k}{C}\right)y = -\frac{w}{C} \qquad (5)$$

This is a second order differential equation with the following standard form, general solution and characteristic equation. (Tuma 1979, p.180):

$$y'' + ay = f(x) \qquad (6)\ \text{(stnd form)}$$

$$y = y_c + y_p \qquad (7)\ \text{(gen sol)}$$

$$\lambda^2 + a = 0 \qquad (8)\ \text{(char eq)}$$

Where

- y_c = complementary solution and
- y_p = particular solution.

For (5):

$$a = -\frac{k}{C} \qquad (9)$$

Substituting (9) into (8) yields:

$$\lambda = \alpha = +/-\sqrt{\frac{k}{C}} \qquad (10)$$

For this method it is appropriate to use a positive α. The complementary solution is:

$$y_c = A_i \cosh(\alpha x) + B_i \sinh(\alpha x) \quad (11)$$

Where:
cosh = hyperbolic cosine
sinh = hyperbolic sine
x = the distance along the roof "beam" from the end wall
A and B = constants determined for a given set of boundary conditions
i = an index which identifies a particular set of boundary conditions

When *f(x)* is a constant as in (5), the particular solution of the equation is that constant divided by *a*:

$$y_p = -\frac{w}{C}\left(\frac{1}{a}\right) = -\frac{w}{C}\left(-\frac{C}{k}\right) = \frac{w}{k} \quad (12)$$

Substituting (11) and (12) into (7) yields the complete solution:

$$y = A_i \cosh(\alpha x) + B_i \sinh(\alpha x) + \frac{w}{k} \quad (13)$$

Appendix I demonstrates that (13) is a solution of (5).

General Solution Roof Diaphragm Shears

The shear at the end walls and each roof bay can be determined by applying a relationship from Timeshenko (p. 202)

$$y' = \frac{v}{C} \quad (14)$$

Where:
v = roof diaphragm shear at any point x
y' = the first derivative of y with respect to x

The first derivative of (13) with respect to x is (Tuma, p.85):

$$y' = A_i \alpha \sinh(\alpha x) + B_i \alpha \cosh(\alpha x) \quad (15)$$

Substituting (15) into (14) and rearranging yields:

$$v = C\alpha \left[A_i \sinh(\alpha x) + B_i \cosh(\alpha x)\right] \quad (16)$$

Application to Post-Frame

Finally, to apply (13) and (16) to a post-frame building, their terms must be converted from a continuous load to a series of concentrated loads. During the first phases of a typical post-frame design, trial values are selected or calculated for the following parameters:

d = the trial embedment depth,
S_f = the trial frame spacing.
b = the effective post width (EP486),
K_p= the lateral stiffness of a post-frame (lbf/in)(EP484, Bohnhoff 1992a),
R_e = the eave support reaction of a propped post-frame (lbf)(EP484, Bohnhoff 1992a)
C_h = the lateral stiffness of the roof diaphragm (lbf/in) (EP484)(Anderson, 1997)

Soil properties are also estimated:

S = the allowable lateral soil bearing pressure per unit of depth (psf/ft) (EP486) and,
n_h = the constant of horizontal soil reaction (psf/ft^2)(EP486).

Values for k and w can be approximated by dividing by the frame spacing. That is:

$$k = \frac{K_p}{S_f} \quad (17)$$

$$w = \frac{R_e}{S_f} \quad (18)$$

C_h is converted to C by multiplying by the frame spacing:

$$C = C_h(S_f) \quad (19)$$

Substituting (17) and (19) into (10) for the positive case yields:

$$\alpha = \frac{1}{S_f}\sqrt{\frac{K_p}{C_h}} \tag{20}$$

Substituting (17) and (18) into (13) yields:

$$y = A_i\cosh(\alpha x) + B_i\sinh(\alpha x) + \frac{R_e}{K_p} \tag{21}$$

Substituting (19) into (16) yields:

$$v = C_h\alpha S_f[A_i\sinh(\alpha x) + B_i\cosh(\alpha x)] \tag{22}$$

Equations (21) and (22) are the general solutions for roof diaphragm deflections and shears in the combined lateral force resistive system, where diaphragm and frame stiffness are constant. To find a particular solution, the coefficients A_i and B_i must be determined by evaluating the appropriate boundary conditions (BC).

BC 1 - RIGID END WALLS

In current practice it is common to consider the end walls as rigid (Figure 2.), the boundary conditions are that $y = 0$ at $x = 0$ and that $y = 0$ at $x = L$. A_1 can be determined by applying the first boundary condition to (21). The cosh(0) = 1 and the sinh(0) = 0, thus:

$$A_1 + \frac{R_e}{K_p} = 0 \tag{23}$$

$$A_1 = -\frac{R_e}{K_p} \tag{24}$$

By substituting (24) into (21), and applying the second boundary condition (and more algebra) B_1 can be determined similarly. (Appendix II presents the derivation of B_3.)

$$B_1 = \frac{R_e}{K_p}\frac{(\cosh(\alpha L)-1)}{\sinh(\alpha L)} \tag{25}$$

Example 1 - Rigid End Walls

This example follows that of Skaggs (1993) presented in the March, 1993, issue of *Frame Building News*. (There are some minor differences in notation.) A post-frame building is 60 ft. wide x 64 ft. long x 12 ft. eave, 3/12 roof pitch. Posts are 6x6 No. 2 Southern Pine at 8 ft. o.c., embedded 4 ft. The windward wall pressure is 16 psf, the windward roof pressure is 4 psf, leeward roof pressure is -14 psf and leeward wall pressure is -10 psf. There are no large end wall openings.

In steps 1 through 3, Skaggs calculated the following values based upon the analog he selected:

$$C_h = 11{,}380 \ pli$$

$$K_p = 149 \ pli \qquad R_e = 2{,}075 \ lbf$$

Calculate compatible deflections

Skaggs used a table of values to find the factor *mD*. This factor is multiplied by R_e to find the actual restraining force provided by the diaphragm. In this case, *mD* = 0.90 and the diaphragm restraining force at this frame was found to equal 2075(.9) = 1868 lbf.

Using the alternate method presented in this paper:

L = 768 in. and S_f = 96 in.

(20) yields:

$$\alpha = \frac{1}{96}\sqrt{\left(\frac{149}{11380}\right)} = \frac{0.001192}{inch}$$

Table 1 column 2 presents the calculated values of the constants for this example.

The maximum roof diaphragm deflection (in.) will be at the center frame (x = 384 in.). Using (21) it is determined to be 1.341 in.

Table 2 columns 2 and 3 presents the deflections calculated at each frame using (21) and DAFI. The restraining force provided by the roof diaphragm at the center frame is 2075 - 149(1.341) = 1875 lbf. This agrees closely with Skaggs estimate of 1868 lbf.

The maximum shear (lbf.) is at the end walls. In this case, (22) reduces to (x = 0, sinh(0) = 0, cosh(0) = 1):

$$Ve_1 = C_h\, S_f\, B_1 \alpha = 7765\ lbf$$

Where:

Ve_1 = maximum shear to end wall, x = 0

The maximum end wall shear calculated by DAFI is 7,774 *lbf*. This is based on 2075 lbf, R_e, applied at each interior frame and 1037.5 lbf, 1/2 R_e, applied to each end frame. Because we are approximating a series of continuous loads with a uniform load, Ve_1 overstates the maximum roof diaphragm shear. The maximum shear in the first bay can be found by evaluating (22) at the center of the first roof bay:

$$Vr_1 = C_h S_f \alpha [A_1 \sinh(\alpha \frac{S_f}{2}) + B_1 \cosh(\alpha \frac{S_f}{2})] = 6740$$

Where:

Vr_1 = shear at the center of the first roof bay.

The shear in the first bay calculated by DAFI is 6,737 lbf. Table 3 columns 2 and 3 present the shears calculated by (22) and DAFI.

The foregoing solution is for rigid end walls. Recent research (Gebremedhin, et. al., 1992, 1993) has shown that end wall deflections can be significant.

BC 2 - NON-RIGID END WALLS OF EQUAL STIFFNESS

In this case the end walls are not rigid, but they are equally stiff. This can be expressed as:

$$y = y_2\ at\ x = 0;\ y = y_2\ at\ x = L$$

The constants in (21) can be determined similarly to the case of rigid end walls. They are:

$$A_2 = y_2 - \frac{R_e}{K_p} \qquad (26)$$

$$B_2 = \frac{(y_2 - \frac{R_e}{K_p})(1 - \cosh(\alpha L))}{\sinh(\alpha L)} \qquad (27)$$

The relationship between end wall shear and deflection can be expressed as:

$$y_2 = \frac{Ve_2}{Kew_2} \qquad (28)$$

Where

Ve_2 = shear at the end walls
Kew_2 = stiffness at the end walls

Substituting (22) and (27) into (28) at $x = 0$, yields:

$$y_2 = \frac{R_e}{K_p[1 - \frac{Kew_2}{\alpha C_h S_f} \frac{\sinh(\alpha L)}{(1 - \cosh(\alpha L))}]} \qquad (29)$$

Example 2 - Non-Rigid End Walls Equal Stiffness

Assume that on the building analyzed by Skaggs both end walls have a stiffness, Kew_2, of 14,225 pli.

Table 1 column 4 presents the calculated values for the constants for this example. Table 2 columns 4 and 5 compares the deflections calculated by (21) and DAFI and Table 3 columns 4 and 5 compare the calculated shears.

BC- 3 NON-RIGID END WALLS OF UNEQUAL STIFFNESS

Another case is where the end walls are not rigid and they are not equally stiff. This can be expressed as:

$$y = y_3 \; at \; x = 0; \; y = y_4 \; at \; x = L$$

The constants in (21) can be determined similarly to the case of rigid end walls. They are:

$$A_3 = y_3 - \frac{R_e}{K_p} \qquad (30)$$

$$B_3 = \frac{(y_4 - \frac{R_e}{K_p}) - (y_3 - \frac{R_e}{K_p})\cosh(\alpha L)}{\sinh(\alpha L)} \qquad (31)$$

Expressions for y_3 and y_4 can be derived similarly to y_2 (29). However, these expressions are complex and somewhat burdensome. This author suggests that acceptable results can be obtained by using estimates of the end wall deflections.

Example 3 - Non-Rigid End Walls of Unequal Stiffness

Assume that on the building analyzed by Skaggs the end wall at x = 0 has a stiffness, Kew_3, of 14,225 pli. Further assume that because of a large opening, the end wall at x = *768* in. has a stiffness, Kew_4, of 7,113 pli. Estimate the deflection of the end walls by using the previously calculated shear, Ve_1 (BC-1).

$$y_3 = \frac{7765}{14225} = 0.546, \; (use \; 0.50)$$

$$y_4 = \frac{7765}{7113} = 1.092, \; (use \; 1.0)$$

Table 1 column 6 presents the calculated values of the constants *A* and *B* corresponding to these estimates. Table 2 columns 6 and 7 compares the calculated deflections and Table 3 columns 6 and 7 compare the shears. These results can be judged to be adequate for design, however, closer agreement to DAFI can be obtained by making a second estimate of the end wall deflections, based on the shears from the first estimate.

$$y_3 = \frac{8107}{14225} = 0.57$$

$$y_4 = \frac{6587}{7113} = 0.93$$

Table 1 column 8 presents the calculated values of the constants *A* and *B* corresponding to these estimates. Table 2 column 8 presents the calculated deflections and Table 3 column 8 presents the calculated shears.

CONCLUSION

In conclusion, it is appropriate to consider the significance of the calculated deflections. Predicting actual eave deflections under a "real life" wind event is a practically impossible task. Therefore this author suggests that the post-frame designer should look upon these deflections as a tool for apportioning load, so that the required strength of each element in the lateral force resistive system may conservatively be found.

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APPENDIX I - EQUATION (13) VERIFICATION

$$y = A_i\cosh(\alpha x) + B_i\sinh(\alpha x) + \frac{w}{k} \qquad \textbf{(13)}$$

Differentiating with respect to x:

$$y' = \alpha(A_i \sinh(\alpha x) + B_i \cosh(\alpha x))$$

Differentiating again with respect to x:

$$y'' = \alpha^2 (A_i \sinh(\alpha x) + B_i \cosh(\alpha x))$$

Substituting: $\alpha^2 = \frac{k}{C}$

$$y'' = \frac{k}{C} (A_i \sinh(\alpha x) + B_i \cosh(\alpha x))$$

Substituting y'' and y into (5):

$$\frac{k}{C}(A_i \sinh(\alpha x) + B_i \cosh(\alpha x)) -$$

$$\frac{k}{C}(A_i \sinh(\alpha x) + B_i \cosh(\alpha x) + \frac{w}{k}) = -\frac{w}{C}$$

Which yields the identity:

$$-\frac{w}{C} = -\frac{w}{C}$$

APPENDIX II - CONSTANT B_3 DERIVATION

$$A_3 = y_3 - \frac{R_e}{K_p} \qquad \textbf{(30)}$$

$$y_4 = A_3 \cosh(\alpha L) + B_3 \sinh(\alpha L) + \frac{R_e}{K_p}$$

Substituting for A_3 yields:

$$y_4 = (y_3 - \frac{R_e}{K_p})\cosh(\alpha L) + B_3 \sinh(\alpha L)$$
$$+ \frac{R_e}{K_p}$$

Rearranging terms yields:

$$B_3 \sinh(\alpha L) = (y_4 - \frac{R_e}{K_p}) - (y_3 - \frac{R_e}{K_p})\cosh(\alpha L)$$

Finally:

$$B_3 = \frac{(y_4 - \frac{R_e}{K_p}) - (y_4 - \frac{R_e}{K_p})\cosh(\alpha L)}{\sinh(\alpha L)}$$

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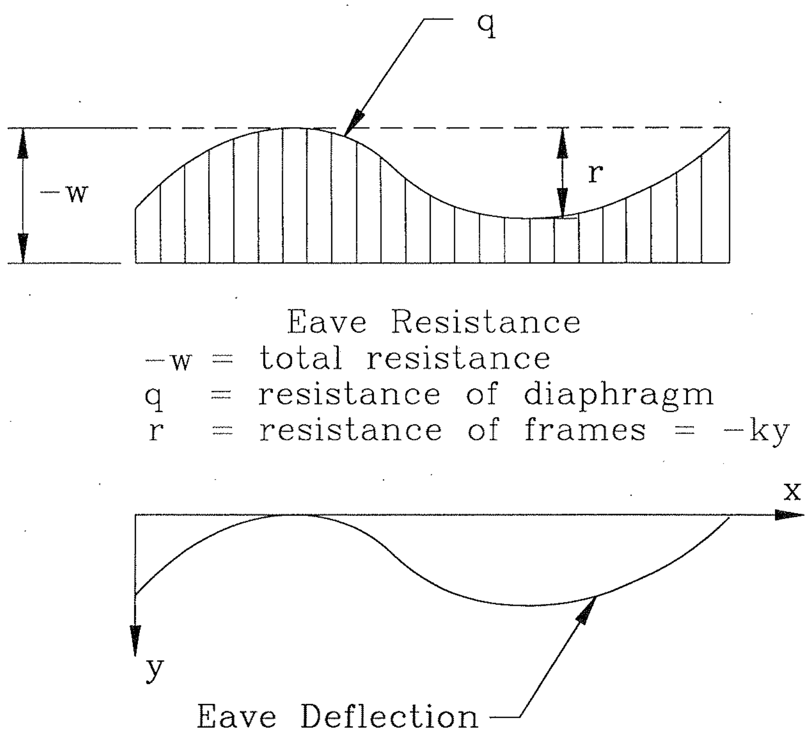

FIGURE 1
General Case

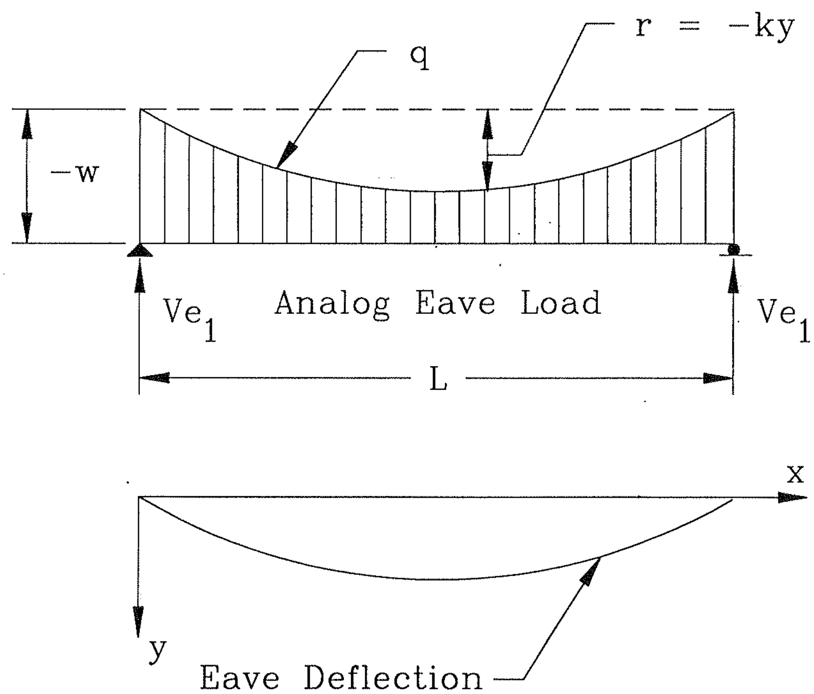

FIGURE 2 -- BOUNDARY CONDITION 1
Rigid End Walls

Table 1 - Values of Constants for Equations (21) and (22) for Boundary Conditions 1, 2 and 3

	BC - 1		BC - 2		BC - 3		
					first estimate		*second estimate*
y, x = 0	0		0.525		0.5		0.57
y, x = 768	0		0.525		1.0		0.93
A	-13.926		-13.401		-13.426		-13.356
B	5.963		5.7348		6.226		6.063

Note: all values are in inches, BC = Boundary Condition

Table 2 - Comparison of Eave Deflections Calculated using Equation (21) vs DAFI

	BC - 1		BC - 2		BC - 3		
Frame Location, x	Eq (21)	DAFI	Eq (21)	DAFI	Eq (21) *first estimate*	DAFI	Eq (21) *second estimate*
0 (EW)	0	0.000	0.525	0.529	0.50 (EST)	0.562	0.57 (EST)
96	0.593	0.592	1.096	1.098	1.126	1.174	1.178
192	1.010	1.009	1.500	1.500	1.584	1.618	1.618
288	1.259	1.258	1.737	1.738	1.881	1.901	1.898
384	1.341	1.340	1.816	1.818	2.019	2.027	2.019
480	1.259	1.258	1.737	1.738	2.002	1.997	1.985
576	1.010	1.009	1.500	1.500	1.828	1.811	1.794
672	0.593	0.592	1.096	1.098	1.495	1.466	1.444
768 (EW)	0.000	0.000	0.525	0.529	1.0 (EST)	0.958	0.93 (EST)

Note: all values are in inches, EW = end wall, EST = estimated

Table 3 - Comparison of Roof Shears Calculated using Equation (22) vs DAFI

	BC - 1		BC - 2		BC - 3		
Roof Panel Location, x	Eq (22)	DAFI	Eq (22)	DAFI	Eq (22) *first estimate*	DAFI	Eq (22) *second estimate*
0(EW)	7765	7774	7472	7519	8107	7996	7894
48	6740	6737	6486	6481	7120	6958	6912
144	4752	4750	4573	4570	5211	5058	5011
240	2827	2825	2720	2718	3371	3224	3176
336	938	938	903	902	1576	1432	1383
432	938	938	903	902	199	341	392
528	2827	2825	2720	2718	1977	2118	2172
624	4752	4750	4573	4570	3781	3923	3981
720	6740	6737	6486	6481	5634	5780	5842
768(EW)	7765	7774	7472	7519	6587	6817	6799

Note: x is in inches, all other values are in pounds, EW = end wall